Question. Let $\{f_n\}$ be a sequence of integrable functions that converges at every point of a cell $K \subset \mathbb{R}^p$ to a function f. Is f integrable on K? Suppose that f is integrable on K, is it true that $\int_K f = \lim \int_K f_n$ K?

Examples.

(a) Let $\mathbb{Q} \cap [0,1] = \{x_n\}_{n=1}^{\infty}$ and f_n be a monotone sequence of integrable functions on $[0,1]$ defined by $f_n(x) = \begin{cases} 1 & \text{if } x \in \{x_1, x_2, \dots, x_n\}, \\ 0 & \text{otherwise.} \end{cases}$

Then the limit function f is defined by $f(x) = \lim f_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$

Note that the convergence of f_n to f is **not uniform** on $[0,1]$, f is **not integrable** on $[0,1]$, and $0 = \lim \displaystyle\int_0^1 f_n \neq \int_0^1 \lim f_n$ since $\int_0^1 f$ does not exist.

(b) Define (discontinuous) f_n and (continuous) f on $K = [0,1]$ by $n \geq 1$ by

$$f_n(x) = \begin{cases} n & \text{if } x \in (0, \frac{1}{n}), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = 0.$$

Note that the convergence of f_n to f is **not uniform** on $[0,1]$, f is Riemann integrable on K, and $1 = \lim \int_0^1 f_n \neq \int_0^1 \lim f_n = \int_0^1 f = 0$.

(c) Let $K = [0,1]$, and (continuous function) f_n be defined for $n \geq 2$ by

$$f_n(x) = \begin{cases} n^2 x & \text{if } x \in [0, \frac{1}{n}], \\ -n^2(x - \frac{2}{n}) & \text{if } x \in [\frac{1}{n}, \frac{2}{n}], \\ 0 & \text{if } x \in [\frac{2}{n}, 1], \end{cases}$$

and (continuous) $f(x) = \lim f_n(x) = 0$ for all $x \in K$. Note that the convergence of f_n to f is **not uniform** on $[0,1]$, f is integrable on K, and $1 = \lim \int_0^1 f_n \neq \int_0^1 \lim f_n = \int_0^1 f = 0$.

These examples indicate that a convergence theorem for the Riemann integral will require some condition in addition to pointwise convergence.

Theorem. Let $\{f_n\}$ be a sequence of integrable functions that converges uniformly on a closed cell $K \subset \mathbb{R}^p$ to a function f. Then f is integrable and $\int_K f = \lim \int_K f_n$.

Proof. Let $\epsilon > 0$ and N be such that $\|f_N - f\|_K < \epsilon$. Now let P_N be a partition of K such that if P, Q are refinements of P_N, then $|S_P(f_N, K) - S_Q(f_N, K)| < \epsilon$, for any choice of the intermediate points. If we use the same intermediate points for f and f_N, then $|S_P(f_N, K) - S_P(f, K)| \leq \|f_N - f\|_K c(K) < \epsilon c(K)$. Since a similar estimate holds for the partition Q, then for refinements P, Q of P_N and corresponding Riemann sums, we have $|S_P(f, K) - S_Q(f, K)| \leq |S_P(f, K) - S_P(f_N, K)| + |S_P(f_N, K) - S_Q(f_N, K)| + |S_Q(f_N, K) - S_Q(f, K)| \leq \epsilon(1 + 2c(K))$. This implies that f is integrable on K.

Since $|\int_K f - \int_K f_n| = |\int_K (f - f_n)| \leq \|f - f_n\|_K c(K)$, we have $\int_K f = \lim \int_K f_n$.

Example. Let $K = [0,1]$, and f_n be defined by

$$f_n(x) = \begin{cases} \sin(n\pi x) & \text{if } x \in [0, \frac{1}{n}], \\ 0 & \text{if } x \in (\frac{1}{n}, 1]. \end{cases}$$

Note that f_n converges to the zero function on $[0,1]$, and the convergence is not uniform on K. However, $\lim \int_0^1 f_n = \lim \dfrac{2}{n\pi} = 0 = \int_0^1 \lim f_n$. This example demonstrates that the uniform convergence is not a necessary condition in the theorem.

Bounded Convergence Theorem. Let $\{f_n\}$ be a sequence of integrable functions on a closed cell $K \subset \mathbb{R}^p$. Suppose that there exists $B > 0$ such that $\|f_n(x)\| \leq B$ for all $n \in \mathbb{N}$, $x \in K$. If the function $f(x) = \lim f_n(x)$, $x \in K$, exists and is integrable, then $\int_K f = \lim \int_K f_n$.

Remark. This theorem has replaced the uniform convergence of f_n by the uniform boundedness of f_n and the integrability of f.

Outline of the Proof. Since $f(x) = \lim f_n(x)$ for $x \in K$ and $\|f_n\|_K \le B$ for all $n \in \mathbb{N}$, there exists M such that $|f(x)| \le M$ and $|f_n(x)| \le M$ for all $x \in K$ and all $n \ge 1$. Since $|f - f_n|$ is integrable on K, there exists a subset $A \subseteq K$ such that $c(K \setminus A) = 0$ and $|f - f_n|$ converges uniformly to 0 on A. This implies that $\int_K f = \lim \int_K f_n$.
To find A, we observe that the convergence of f_n to f is not uniform on K if there exists $\epsilon > 0$ such that the set $A_n = \{x \in K \mid \exists\, j \ge n \text{ such that } |f_j(x) - f(x)| \ge \epsilon\} \ne \emptyset$. Note that

(a) $\{A_n\}$ is a nested sequence, i.e. $A_1 \supseteq A_2 \supseteq \cdots A_n \supseteq A_{n+1} \supseteq \cdots$,

(b) $\displaystyle\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n = \emptyset$ and $\lim c(A_n) = 0$,

(c) $A_n \ne \emptyset$ for any $\delta < \epsilon$,

(d) f_n converges uniformly to f on each $K \setminus A_j$, $j \in \mathbb{N}$.

Examples. Use suitable convergence theorem to prove the following.

(a) If $a > 0$, then $\displaystyle\lim_n \int_0^a e^{-nx}\,dx = 0$.

(b) If $0 < a < 2$, then $\displaystyle\lim_n \int_a^2 e^{-nx^2}\,dx = 0$. What happens if $a = 0$?

(c) If $a > 0$, then $\displaystyle\lim_n \int_a^{\pi} \frac{\sin nx}{nx}\,dx = 0$. What happens if $a = 0$?

(d) Let $f_n(x) = \dfrac{nx}{1+nx}$ for $x \in [0,1]$, and let $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0,1]. \end{cases}$
Then $\lim f_n(x) = f(x)$ for all $x \in [0,1]$ and that $\lim \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$.

(e) Let $h_n(x) = nxe^{-nx^2}$ for $x \in [0,1]$ and let $h(x) = 0$. Then $0 = \int_0^1 h(x)\,dx \ne \lim \int_0^1 h_n(x)\,dx = \dfrac{1}{2}$.

Monotone Convergence Theorem. Let $\{f_n\}$ be a monotone sequence of integrable functions on a closed cell $K \subset \mathbb{R}^p$. If the function $f(x) = \lim f_n(x)$, $x \in K$, exists and is integrable, then $\int_K f = \lim \int_K f_n$.

Proof. Suppose that $f_1(x) \le f_2(x) \le \cdots \le f(x)$ for all $x \in K$, then $f_n(x) \in [f_1(x), f(x)]$ for all $n \in \mathbb{N}$ and $\|f_n(x)\| \le |f_1(x)| + |f(x)| \le \sup_{x\in K} |f_1(x)| + \sup_{x\in K} |f(x)| = B$ for all $x \in K$ and for all $n \in \mathbb{N}$, so we can apply the Bounded Convergence Theorem to establish that $\int_K f = \lim \int_K f_n$.

Remark. Note that the convergence theorem may fail if K in not compact.

Example. Let $f_n(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \in [1,n] \\ 0 & \text{if } x > n. \end{cases}$ Then each f_n is integrable on $[1,\infty)$, and $\{f_n\}$ is a bounded, monotone sequence that converges uniformly on $[1,\infty)$ to a continuous function $f(x) = 1/x$. Note that $\lim \int_1^\infty f_n \ne \int_1^\infty \lim f_n$ since f is not integrable on $[1,\infty)$.

Example. Let $g_n(x) = \begin{cases} \dfrac{1}{n} & \text{if } x \in [0,n^2] \\ 0 & \text{if } x > n^2. \end{cases}$ Then each g_n is integrable on $[1,\infty)$, and $\{g_n\}$ is a bounded, monotone sequence that converges $[1,\infty)$ to an integrable function $g(x) \equiv 0$. Note that $\lim \int_1^\infty g_n \ne \int_1^\infty \lim g_n$.

Definition. If $\{f_n\}$ is a sequence of functions defined on a subset D of $\mathbb{R}^p$ with values in $\mathbb{R}^q$, the sequence of partial sums (s_n) of the series $\sum f_n$ is defined for x in D by $s_n(x) = \displaystyle\sum_{j=1}^{n} f_j(x)$. In case the sequence $\{s_n\}$ **converges** on D to a function f, we say that the infinite series of functions $\sum f_n$

converges to f on D. If the sequence $\{s_n\}$**converges uniformly** on D to a function f, we say that the infinite series of functions $\sum f_n$ **converges uniformly** to f on D.
Remark (Cauchy Criterion). It is easy to see that $\sum f_k$ converges uniformly on D if and only if for each $\epsilon > 0$, there exists $M = M(\epsilon) \in \mathbb{N}$ such that for any $n, m \geq M$ and any $x \in D$, we have $\|s_n(x) - s_m(x)\| < \epsilon$.
Dirichlet's Test Let $\{f_n\}$ be a sequence of functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ such that the partial sums $s_n = \displaystyle\sum_{j=1}^{n} f_j$, $n \in \mathbb{N}$, are all bounded. Let $\{\phi_n\}$ be a decreasing sequence of functions on D to $\mathbb{R}$ which converges uniformly on D to zero. Then the series $\displaystyle\sum_{n=1}^{\infty}(f_n\phi_n)$ converges uniformly on D.

Abel's Test Let $\displaystyle\sum_{n=1}^{\infty} f_n$ be a series of functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ which is uniformly convergent on D. Let $\{\phi_n\}$ be a sequence of functions on D to $\mathbb{R}$ which is bounded on D. Then the series $\displaystyle\sum_{n=1}^{\infty}(f_n\phi_n)$ converges uniformly on D.

Outline of the proofs. For Dirichlet's test, observe that $|\displaystyle\sum_{j=n}^{m} \phi_j f_j| = |\phi_{m+1}s_m - \phi_n s_{n-1} + \sum_{j=n}^{m}(\phi_j - \phi_{j+1}s_j|$. For Abel's test, if $|\phi_j(x)| \leq B$ for all $j \in \mathbb{N}$ and for all $x \in D$, then $|\displaystyle\sum_{j=n}^{m} \phi_j f_j| \leq B \sum_{j=n}^{m} |f_j|$.

Theorem. If f_n is continuous on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then f is continuous on D.
Term-by-Term Integration Theorem. Suppose that the real-valued functions f_n, $n \in \mathbb{N}$, are integrable on $K = [a, b]$. If the series $\sum f_n$ converges to f uniformly on K, then f is integrable on K and $\displaystyle\int_K f = \sum_{j=1}^{\infty} \int_K f_n$.
Term-by-Term Differentiation Theorem. For each $n \in \mathbb{N}$, let f_n be a real-valued function on $K = [a, b]$ which has a derivative f_n' on K. Suppose that the infinite series $\sum f_n$ converges for at least one point of K and that the series of derivatives $\sum f_n'$ converges uniformly on K. Then there exists a real-valued function f on K such that $\sum f_n$ converges uniformly on K to f. In addition, f has a derivative on K and $f' = \sum f_n'$.
Proof. Suppose that the partial sum s_n of $\sum f_n$ converges at $x_0 \in K$. For each $x \in K$ and any $m, n \in \mathbb{N}$, by the Mean Value Theorem, the equality $s_m(x) - s_n(x) = s_m(x_0) - s_n(x_0) + (x - x_0)(s_m'(y) - s_n'(y))$ holds for some y lying between x and x_0. The uniform convergence of $\sum f_n'$ and the convergence of $\sum f_n(x_0)$ lead to the uniform convergence of $\sum f_n$ on K.
Suppose that $\sum f_n'$ converges uniformly to g on K. For each $x, c \in K$ and any $m, n \in \mathbb{N}$, by the Mean Value Theorem, the equality $s_m(x) - s_n(x) = s_m(c) - s_n(c) + (x - c)(s_m'(y) - s_n'(y))$ holds for some y lying between x and c. We infer that, when $x \neq c$, then $|\dfrac{s_m(x) - s_m(c)}{x - c} - \dfrac{s_n(x) - s_n(c)}{x - c}| \leq \|s_m' - s_n'\|_K$. Given $\epsilon > 0$, by the uniform convergence of $\sum f_n'$, there exists a $M(\epsilon)$ such that if $m, n \geq M(\epsilon)$ then $\|s_m' - s_n'\|_K < \epsilon$. Taking the limit with respect to m, we get $|\dfrac{f(x) - f(c)}{x - c} - \dfrac{s_n(x) - s_n(c)}{x - c}| \leq \epsilon$ when $n \geq M(\epsilon)$. Since $g(c) = \lim s_n'(c)$, there exists an $N(\epsilon)$ such that if $n \geq N(\epsilon)$, then $|s_n'(c) - g(c)| < \epsilon$. Now let $L = \max\{M(\epsilon), N(\epsilon)\}$. In view of the existence of $s_L'(c)$, if $0 < |x - c| < \delta_{L(\epsilon)}$, then $|\dfrac{s_L(x) - s_L(c)}{x - c} - s_L'(c)| < \epsilon$. Therefore, it follows that if $0 < |x - c| < \delta_{L(\epsilon)}$, then $|\dfrac{f(x) - f(c)}{x - c} - g(c)| < 3\epsilon$. This shows that $f'(c)$ exists and equals $g(c)$.

Example (a). For each $k \in \mathbb{N}$ and for each $x \in [-1, 1]$, define $f_k(x) = \dfrac{x^k}{k^2}$. Then $\displaystyle\sum_{k=1}^{\infty} f_k$ converges uniformly on $[-1,1]$, and $\sum_{k=1}^{\infty} f_k' = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k}$ converges uniformly on any $[-r,r]$, where $0 \le r < 1$.

Example (b). For each $k \ge 0$ and for each $x \in (-1,1)$, define $f_k(x) = (-1)^k x^k$. Then $\sum_{k=0}^{\infty} f_k$ converges to $f(x) = \frac{1}{1+x}$ uniformly on any $[-r,r]$, where $0 \le r < 1$.

Example (c). Using (a), one observes that $\sum_{k=0}^{\infty} f_k' = \sum_{k=0}^{\infty} (-1)^k x^{2k}$ converges to $\frac{1}{1+x^2}$ on $(-1,1)$ and it is not convergent at $x = \pm 1$, while $\sum_{k=0}^{\infty} f_k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ converges to $\tan^{-1} x$ on $(-1,1)$ uniformly on $[-1,1]$.

Example (d). Note that $\sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ converges uniformly on $\mathbb{R}$ by Cauchy Criterion and the criterion is not applicable for $\sum_{k=1}^{\infty} f_k' = \sum_{k=1}^{\infty} \frac{\cos kx}{k}$ since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Definition. Let f be defined on $[a,\infty) \times [\alpha,\beta]$ to $\mathbb{R}$. Suppose that for each $t \in J = [\alpha,\beta]$ the infinite integral $F(t) = \int_a^\infty f(x,t)dx = \lim_{c\to\infty} \int_a^c f(x,t)dx$ exists. We say that this convergence is uniform on J if for every $\epsilon > 0$ there exists a number $M(\epsilon)$ such that if $c \ge M(\epsilon)$ and $t \in J$, then $|F(t) - \int_a^c f(x,t)dx| < \epsilon$.

Dominated Convergence Theorem. Suppose that f is integrable over $[a,c]$ for all $c \ge a$ and all $t \in J = [\alpha,\beta]$. Suppose that there exists a positive function ϕ defined for $x \ge a$ such that $|f(x,t)| \le \phi(x)$ for $x \ge a$, $t \in J$, and such that the integral $\int_a^\infty \phi(x)dx$ exists. Then, for each $t \in J$, the integral $F(t) = \int_a^\infty f(x,t)dx$ is (absolutely) convergent and the convergence is uniform on J.

Dirichlet's Test Let f be continuous on $[a,\infty) \times [\alpha,\beta]$ and suppose that there exists a constant A such that $|\int_a^c f(x,t)dx| \le A$ for $c \ge a$, $t \in J = [\alpha,\beta]$. Suppose that for each $t \in J$, the function $\phi(x,t)$ is monotone decreasing for $x \ge a$, and converges to 0 as $x \to \infty$ uniformly for $t \in J$. Then the integral $F(t) = \int_a^\infty f(x,t)\phi(x,t)dx$ converges uniformly on J.

Examples.

(a) $\int_0^\infty \frac{dx}{x^2+t^2}$ converges uniformly for $|t| \ge a > 0$.

(b) $\int_0^\infty \frac{dx}{x^2+t}$ converges uniformly for $t \ge a > 0$ and diverges when $t \le 0$.

(c) $\int_0^\infty e^{-x} \cos tx dx$ converges uniformly for $t \in \mathbb{R}$ by the dominated convergence theorem.

(d) $\int_0^\infty e^{-x^2-t^2/x^2} dx$ converges uniformly for $t \in \mathbb{R}$ by the dominated convergence theorem.

Theorem. Let f be continuous on $K = [a,b] \times [c,d]$ to $\mathbb{R}$ and $F : [c,d] \to \mathbb{R}$ be defined by $F(t) = \int_a^b f(x,t)dx$. Then F is continuous on $[c,d]$ to $\mathbb{R}$.

Proof. Let $\epsilon > 0$, since f is uniformly continuous on K, there exists a $\delta(\epsilon) > 0$ such that if t and t_0 belong to $[c,d]$ and $|t - t_0| < \delta(\epsilon)$, then $|f(x,t) - f(x,t_0)| < \epsilon$, for all $x \in [a,b]$. It follows that $|F(t) - F(t_0)| = |\int_a^b \big(f(x,t) - f(x,t_0)\big)dx| \le \int_a^b |f(x,t) - f(x,t_0)|dx \le \epsilon(b-a)$, which establishes the continuity of F.

Remark. Suppose that f is continuous on $[a,\infty) \times [c,d]$ to $\mathbb{R}$ and $F(t) = \int_a^\infty f(x,t)dx$ converges uniformly on $[c,d]$, we let $F_n(t) = \int_a^{a+n} f(x,t)dx$. Then F_n is continuous on $[c,d]$ and F is continuous on $[c,d]$ since F_n converges to F uniformly on $[c,d]$.

Theorem. Let f and its partial derivative f_t be continuous on $K = [a,b] \times [c,d]$ to $\mathbb{R}$. Then the function $F(t) = \int_a^b f(x,t)dx$ is differentiable on (c,d) and $F'(t) = \int_a^b f_t(x,t)dx$ for $t \in (c,d)$.

Proof. From the uniform continuity of f_t on K we infer that if $\epsilon > 0$, then there is a $\delta(\epsilon) > 0$ such that if $|t - t_0| < \delta(\epsilon)$, then $|f_t(x,t) - f_t(x,t_0)| < \epsilon$ for all $x \in [a,b]$. Let t, t_0 satisfy this condition and apply the Mean Value Theorem to obtain a t_1 (which may depend on x and lies between t and t_0) such that $f(x,t) - f(x,t_0) = (t - t_0)f_t(x,t_1)$. Combining these two relations, we infer that if $0 < |t - t_0| < \delta(\epsilon)$, then $|\dfrac{f(x,t) - f(x,t_0)}{t - t_0} - f_t(x,t_0)| < \epsilon$ for all $x \in [a,b]$. Thus, we obtain $|\dfrac{F(t) - F(t_0)}{t - t_0} - \int_a^b f_t(x,t_0)dx| \leq \int_a^b |\dfrac{f(x,t) - f(x,t_0)}{t - t_0} - f_t(x,t_0)|dx \leq \epsilon(b-a)$, which establishes $F'(t) = \int_a^b f_t(x,t)dx$.

Generalization. Let S be a measurable subset of $\mathbb{R}^n$ and T a subset of $\mathbb{R}^m$. Suppose $f(x,y)$ is a function on $T \times S$ that is integrable as a function of $y \in S$ for each $x \in T$, and let F be defined on T by $F(x) = \int_S f(x,y)dy$ for $x \in T$.

(a) If $f(x,y)$ is continuous as a function of $x \in T$ for each $y \in S$, and there exists a constant C such that $|f(x,y)| \leq C$ for all $x \in T$ and $y \in S$, then F is continuous on T.

(b) Suppose T is open. If $f(x,y)$ is of class C^1 as a function of $x \in T$ for each $y \in S$, and there is a constant C such that $|\nabla_x f(x,y)| \leq C$ for all $x \in T$ and $y \in S$, then F is of class C^1 on T and $\dfrac{\partial F}{\partial x_j}(x) = \int_S \dfrac{\partial f}{\partial x_j}(x,y)dy$ for $x \in T$.

Proof. Let $\{x_j\}$ be a sequence in T converging to $x \in T$. For each $j \in \mathbb{N}$ and $y \in S$, let $f_j(y) = f(x_j, y)$ and let $f(y) = f(x,y)$. Then each f_j and f are integrable on S, and $|f_j(y)| \leq C$ and $f_j(y) \to f(y)$ for all j and all $y \in S$. The bounded convergence theorem implies that $\lim F(x_j) = \lim \int_S f(x_j,y)dy = \lim \int_S f_j = \int_S \lim f_j = \int_S \lim f(x_j,y) = \int_S f(x,y) = F(x)$. Hence, F is continuous and this proves (a).

Part (b) is proved by applying the bounded convergence theorem to the sequence of difference quotients $\dfrac{f(x + h_j e_i, y) - f(x,y)}{h_j}$, where e_i denotes the unit vector in the x_i-coordinate and $\{h_j\}$ is a sequence of numbers tending to zero. The uniform bound on these quotients is obtained by applying the mean value theorem .

Examples.

(a) Let $f(x,t) = \dfrac{\cos tx}{1+x^2}$ for $x \in [0,\infty)$ and $t \in (-\infty,\infty)$. Then $\int_0^\infty f(x,t)$ converges uniformly for $t \in \mathbb{R}$ by Dominated Convergence Theorem.

(b) Let $f(x,t) = e^{-x}x^t$ for $x \in [0,\infty)$ and $t \in [0,\infty)$. For any $\beta > 0$, the integral $\int_0^\infty f(x,t)$ converges uniformly for $t \in [0,\beta]$ by Dominated Convergence Theorem. Similarly, the Laplace transform of x^n, $n = 0,1,2,\ldots$, defined by $\mathscr{L}\{x^n\}(t) = \int_0^\infty x^n e^{-tx}dx$ also converges uniformly for $t \geq \gamma > 0$ to $\dfrac{n!}{t^{n+1}}$. For $t \geq 1$, define the gamma function Γ by $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$. Then it is uniformly convergent on an interval containing t. Note that $\Gamma(t+1) = t\Gamma(t)$ and hence $\Gamma(n+1) = n!$ for any $n \in \mathbb{N}$.

(c) Let $f(x,t) = e^{-tx}\sin x$ for $x \in [0,\infty)$ and $t \geq \gamma > 0$. Then the integral $F(t) = \int_0^\infty e^{-tx}\sin x dx$ is converges uniformly for $t \geq \gamma > 0$ by Dominated Convergence Theorem and it is called the laplace transform of $\sin x$, denoted by $\mathscr{L}\{\sin x\}(t)$. Note that an elementary calculation shows that $\mathscr{L}\{\sin x\}(t) = \dfrac{1}{1+t^2}$.

(d) Let $f(x,t,u) = e^{-tx}\dfrac{\sin ux}{x}$ for $x \in [0,\infty)$ and $t,u \in [0,\infty)$. By taking $\phi = e^{-tx}/x$ and by applying the Dirichlet's test, one can show that $\int_\gamma^\infty f(x,t,u)$ converges uniformly for $t \geq \gamma \geq 0$. Note that if $F(t,u) = \mathscr{L}\{\dfrac{\sin ux}{x}\}(t) = \int_0^\infty e^{-tx}\dfrac{\sin ux}{x}dx$, then $\dfrac{\partial F}{\partial u}(t,u) = \int_0^\infty e^{-tx}\cos uxdx = \dfrac{t}{t^2+u^2}$ and $F(t,u) = \tan^{-1}\dfrac{u}{t}$. By setting $u = 1$ and by letting $t \to 0^+$, we obtain that $\int_0^\infty \dfrac{\sin x}{x}dx = \dfrac{\pi}{2}$.

(e) Let $G(t) = \int_0^\infty e^{-x^2-t^2/x^2}dx$ for $t > 0$. Then $G'(t) = -2G(t)$ and $G(t) = \dfrac{\sqrt{\pi}}{2}e^{-2t}$.

(f) Let $F(t) = \int_0^\infty e^{-x^2}\cos tx dx$ for $t \in \mathbb{R}$. Then $F'(t) = -\dfrac{t}{2}F(t)$ and $F(t) = \dfrac{\sqrt{\pi}}{2}e^{-t^2/4}$.

Leibiniz's Formula. Let f and its partial derivative f_t be continuous on $K = [a,b] \times [c,d]$ to $\mathbb{R}$ and α and β be differentiable functions on $[c,d]$ and have values in $[a,b]$. Then the function $\phi(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)dx$ is differentiable on (c,d) and $\phi'(t) = f(\beta(t),t)\beta'(t) - f(\alpha(t),t)\alpha'(t) + \int_{\alpha(t)}^{\beta(t)} f_t(x,t)dx$ for $t \in (c,d)$.

Proof. Let H be defined for (u,v,t) by $H(u,v,t) = \int_v^u f(x,t)dx$, where u,v belong to $[a,b]$ and t belongs to $[c,d]$. Then $\phi(t) = H(\beta(t),\alpha(t),t)$. Applying the Chain Rule to obtain the result.

Interchange Theorem. Let f be continuous on $K = [a,b] \times [c,d]$ to $\mathbb{R}$.
Then $\int_c^d \left\{\int_a^b f(x,t)dx\right\} dt = \int_a^b \left\{\int_c^d f(x,t)dt\right\} dx$.

Proof. Since f is uniformly continuous on K, if $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $|x-x'| < \delta(\epsilon)$ and $|t-t'| < \delta(\epsilon)$, then $|f(x,t) - f(x',t')| < \epsilon$. Let n be chosen so large that $(b-a)/n < \delta(\epsilon)$ and $(d-c)/n < \delta(\epsilon)$ and divide K into n^2 equal rectangles by dividing $[a,b]$ and $[c,d]$ each into n equal parts. For $j = 0,1,\ldots,n$, we let $x_j = a + (b-a)j/n,\ t_j = c + (d-c)j/n$.

Then $\displaystyle\int_c^d \left\{\int_a^b f(x,t)dx\right\} dt = \sum_{i=1}^n\sum_{j=1}^n \int_{t_{i-1}}^{t_i} \left\{\int_{x_{j-1}}^{x_j} f(x,t)dx\right\} dt = \sum_{i=1}^n\sum_{j=1}^n f(x_j',t_i')(x_j - x_{j-1})(t_i - t_{i-1})$.

Similarly, $\displaystyle\int_a^b \left\{\int_c^d f(x,t)dt\right\} dx = \sum_{i=1}^n\sum_{j=1}^n f(x_j'',t_i'')(x_j - x_{j-1})(t_i - t_{i-1})$.

The uniform continuity of f implies that two iterated integrals differ by at most $\epsilon(b-a)(d-c)$. Since $\epsilon > 0$ is arbitrary, the equality of these integrals is confirmed.

Example. Let $A \subseteq \mathbb{R}^2$ be the set consisting of all pairs $(i/p, j/p)$ where p is a prime number, and $i,j = 1,2,\ldots,p-1$. (a) Show that each horizontal and each vertical line in $\mathbb{R}^2$ intersects A in a finite number (often zero) of points and that A does not have content. Let f be defined on $K = [0,1]\times[0,1]$ by $f(x,y) = 1$ for $(x,y) \in A$ and $f(x,y) = 0$ otherwise. (b) Show that f is not integrable on K. However, the iterated integrals exist and satisfy $\int_0^1 \left\{\int_0^1 f(x,y)dx\right\} dy = \int_0^1 \left\{\int_0^1 f(x,y)dy\right\} dx$.

Example. Let $K = [0,1] \times [0,1]$ and let $f : K \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if either } x \text{ or } y \text{ is irrational,} \\ \dfrac{1}{n} & \text{if } y \text{ is rational and } x = \dfrac{m}{n} \text{ where } m \text{ and } n > 0 \text{ are relatively prime integers.} \end{cases}$$

Show that $\int_K f = \int_0^1 \left\{\int_0^1 f(x,y)dx\right\} dy = 0$, but that $\int_0^1 f(x,y)dy$ does not exist for rational x.

Fubini's Theorem. Let f be continuous on $K = [a,b]\times[c,d]$ to $\mathbb{R}$. Then $\int_K f = \int_c^d \left\{\int_a^b f(x,y)dx\right\} dy = \int_a^b \left\{\int_c^d f(x,y)dy\right\} dx$.

Proof. Let F be defined for $y \in [c,d]$ by $F(y) = \int_a^b f(x,y)dx$. Let $c = y_0 < y_1 < \cdots < y_r = d$ be a partition of $[c,d]$, let $a = x_0 < x_1 < \cdots < x_s = b$ be a partition of $[a,b]$, and let P denote the partition of K obtained by using the cells $[x_{i-1},x_i] \times [y_{j-1},y_j]$. Let y_j^* be any point in $[y_{j-1},y_j]$ and note that $F(y_j^*) = \displaystyle\int_a^b f(x,y_j^*)dx = \sum_{i=1}^s \int_{x_{i-1}}^{x_i} f(x,y_j^*)dx$. The Mean Value Theorem implies that for each j, there exists a $x_{ji}^* \in [x_{i-1},x_i]$ such that $F(y_j^*) = \displaystyle\sum_{i=1}^s f(x_{ji}^*,y_j^*)(x_i - x_{i-1})$. We multiply by $(y_j - y_{j-1})$ and add to obtain $\displaystyle\sum_{j=1}^r F(y_j^*)(y_j - y_{j-1}) = \sum_{j=1}^r\sum_{i=1}^s f(x_{ji}^*,y_j^*)(x_i - x_{i-1})(y_j - y_{j-1})$. We have shown that an arbitrary Riemann sum for F on $[c,d]$ is equal to a particular Riemann sum of f on

K corresponding to the partition P. Since f is integrable on K, the existence of the iterated integral and its equality with the integral on K is established.
A minor modification of the proof given for the preceding theorem yields the following, slightly stronger, assertion.
Generalization Theorem. Let f be integrable on $K = [a, b] \times [c, d]$ to $\mathbb{R}$ and suppose that for each $y \in [c, d]$, the function $x \mapsto f(x, y)$ of $[a, b]$ into $\mathbb{R}$ is continuous except possibly for a finite number of points, at which it has one-sided limits. Then $\int_K f = \int_c^d \left\{ \int_a^b f(x, y)dx \right\} dy$.
Corollary. Let $A \subseteq \mathbb{R}^2$ be given by $A = \{(x, y) : \alpha(y) \le x \le \beta(y),\ c \le y \le d\}$, where α and β are continuous functions on $[c, d]$ with values in $[a, b]$. If f is continuous on $A \mapsto \mathbb{R}$, then f is integrable on A and $\int_A f = \int_c^d \left\{ \int_{\alpha(y)}^{\beta(y)} f(x, y)dx \right\} dy$.
Proof. Let $K \supseteq A$ be a closed cell and f_K be the extension of f to K. Since ∂A has content zero, f_K is integrable on K. Now for each $y \in [c, d]$ the function $x \mapsto f_K(x, y)$ is continuous except possibly at the two points $\alpha(y)$ and $\beta(y)$, at which it has one-sided limits. It follows from the preceding theorem that $\int_A f = \int_K f_K = \int_c^d \left\{ \int_a^b f_K(x, y)dx \right\} dy = \int_c^d \left\{ \int_{\alpha(y)}^{\beta(y)} f(x, y)dx \right\} dy$.
Example. Let R denote the triangular region in the first quadrant bounded by the lines $y = x$, $y = 0$, and $x = 1$. Then $\int_0^1 \int_y^1 \dfrac{\sin x}{x} dxdy = \int_0^1 \int_0^x \dfrac{\sin x}{x} dydx$.
Example. $\displaystyle\int_0^2 \int_{y/2}^1 ye^{-x^3}\ dxdy = \int_0^1 \int_0^{2x} ye^{-x^3}\ dydx = \int_0^1 \frac{y^2}{2} e^{-x^3}|_0^{2x}\ dx$
$\displaystyle= \int_0^1 2x^2 e^{-x^3}\ dx = \frac{-2e^{-x^3}}{3}|_0^1 = \frac{2}{3}(1 - e^{-1})$.
Example. For $\beta > \alpha > 0$, let $R = [0, \infty) \times [\alpha, \beta]$ and $f(x, t) = e^{-tx}$. Then $\log \dfrac{\beta}{\alpha} = \int_\alpha^\beta \dfrac{1}{t} dt =$ $\int_\alpha^\beta \int_0^\infty e^{-tx} dxdt = \int_0^\infty \int_\alpha^\beta e^{-tx} dtdx = \int_0^\infty \dfrac{e^{-\alpha x} - e^{-\beta x}}{x} dx$.
Lemma. Let $\Omega \subseteq \mathbb{R}^p$ be open, $\phi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$, and A be a bounded set with $\mathrm{Cl}(A) = \bar{A} \subset \Omega$.
Then there exists a bounded open set Ω_1 with

$$\bar{A} \subset \Omega_1 \subset \bar{\Omega}_1 \subseteq \Omega$$

and a constant $M > 0$ such that if A is contained in the union of a finite number of closed cells in Ω, with total content at most α, then $\phi(A)$ is contained in the union of a finite number of closed cells in Ω, with total content at most $(\sqrt{p}M)^p \alpha$.
Proof. If $\Omega = \mathbb{R}^p$, let $\delta = 1$; otherwise let $\delta = \dfrac{1}{2} \inf\{\|a - x\| : a \in \bar{A},\ x \notin \Omega\} > 0$.
Let

$$\Omega_1 = \{y \in \mathbb{R}^p : \|y - a\| < \delta \text{ for some } a \in A\}$$

$$M = \sup\{\|d\phi(x)\|_{pp} = \sup_{0 \ne v \in \mathbb{R}^p} \|d\phi_x(v)\|/\|v\| : x \in \Omega_1\} < \infty.$$

If $A \subseteq I_1 \cup \cdots \cup I_m$, where the I_j are closed cells contained in Ω_1, then it follows that for $x, y \in I_j$ we have

$$\|\phi(x) - \phi(y)\| \le M\|x - y\|.$$

Suppose the side length of I_j is $2r_j$ and take x to be the center of I_j; then if $y \in I_j$, we have

$$\|x - y\| \le \sqrt{p} r_j.$$

Thus $\phi(I_j)$ is contained in a closed cell of side length $2\sqrt{p}Mr_j$, and $\phi(A)$ is contained in the union of a finite number closed cells with total content at most $(\sqrt{p}M)^p \alpha$.
Theorem. Let $\Omega \subseteq \mathbb{R}^p$ be open, $\phi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$. If $A \subset \Omega$ has content zero and if $\bar{A} \subset \Omega$, then $\phi(A)$ has content zero.

Proof. Apply the lemma for arbitrary $\alpha > 0$.
Corollary. Let $r < p$, $\Omega \subseteq \mathbb{R}^r$ be open, and $\psi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$. If $A \subset \Omega$ is a bounded set with $\bar{A} \subset \Omega$, then $\psi(A)$ has content zero in $\mathbb{R}^p$.
Proof. Let $\Omega_0 = \Omega \times \mathbb{R}^{p-r}$. Then Ω_0 is open in $\mathbb{R}^p$.
Define $\phi : \Omega_0 \to \mathbb{R}^p$ by

$$\phi(x_1, \ldots, x_r, x_{r+1}, \ldots, x_p) = \psi(x_1, \ldots, x_r).$$

Thus $\phi \in C^1(\Omega_0)$.
Let $A_0 = A \times \{(0, \ldots, 0)\}$. Then $\bar{A}_0 \subset \Omega_0$ and A_0 has content zero in $\mathbb{R}^p$.
It follows that $\psi(A) = \phi(A_0)$ has content zero in $\mathbb{R}^p$.
Theorem. Let $\Omega \subseteq \mathbb{R}^p$ be open, $\phi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$. Suppose that A has content, $\bar{A} \subset \Omega$, and the Jacobian determinant $J_\phi(x) = \det(d\phi)(x) \neq 0$ for all $x \in \mathrm{Int}(A)$. Then $\phi(A)$ has content.
Proof. Since $\phi(\bar{A})$ is compact and $\phi(A) \subseteq \phi(\bar{A})$, $\phi(A)$ is bounded in $\mathbb{R}^p$, and $\overline{\phi(A)} \subseteq \phi(\bar{A})$.
Now $\partial\phi(A) \cup \mathrm{Int}(\phi(A)) = \overline{\phi(A)} \subseteq \phi(\bar{A}) = \phi(\mathrm{Int}(A) \cup \partial A)$,
and since $\phi \in C^1(\Omega)$ and $J_\phi(x) \neq 0$ for all $x \in \mathrm{Int}(A)$, we conclude that $\phi(\mathrm{Int}(A)) \subseteq \mathrm{Int}(\phi(A))$ by the inverse function theorem.
Hence, we infer that $\partial\phi(A) \subseteq \phi(\partial A)$, and $c(\partial\phi(A)) = 0$.
Thus $\phi(A)$ has content.
Corollary. Let $\Omega \subseteq \mathbb{R}^p$ be open, $\phi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$ and be injective on Ω.
If A has content, $\bar{A} \subset \Omega$, and $J_\phi(x) \neq 0$ for all $x \in \mathrm{Int}(A)$.
Then $\partial\phi(A) = \phi(\partial A)$.
Proof. The proof of the inclusion $\phi(\partial A) \supseteq \partial\phi(A)$ is given in the proof of the proceeding theorem.
To establish the identity $\partial\phi(A) = \phi(\partial A)$, we only need to show the that $\phi(\partial A) \subseteq \partial\phi(A)$.
Let $x \in \partial A$, then there exists a sequence $\{x_n\}$ in A and a sequence $\{y_n\}$ in $\Omega \setminus A$, such that $\lim\limits_{n\to\infty} x_n = x = \lim\limits_{n\to\infty} y_n$.
Since ϕ is continuous, we have $\lim\limits_{n\to\infty} \phi(x_n) = \phi(x) = \lim\limits_{n\to\infty} \phi(y_n)$.
On the other hand, since ϕ is injective on Ω, $\phi(y_n) \notin \phi(A)$.
Thus $\phi(x) \in \partial\phi(A)$ which implies that $\phi(\partial A) \subseteq \partial\phi(A)$.
Theorem. Let

$$\begin{aligned} L \in \mathscr{L}(\mathbb{R}^p) &= \{L : \mathbb{R}^p \to \mathbb{R}^p \mid L = (l_{ij}) \text{ is an } p \times p \text{ matrix over } \mathbb{R}\} \\ &= \text{the space of linear mappings on } \mathbb{R}^p, \end{aligned}$$

and let $A \in \mathscr{D}(\mathbb{R}^p)$.
Then $c(L(A)) = |\det L|\, c(A)$.
Outline of the Proof. If $\det L = 0$, then $L(\mathbb{R}^p) = \mathbb{R}^r$ for some $r < p$, and that $c(L(A)) = 0$ for all $A \in \mathscr{D}(\mathbb{R}^p)$.
If $\det L \neq 0$, and if $A \in \mathscr{D}(\mathbb{R}^p)$, then $L(A) \in \mathscr{D}(\mathbb{R}^p)$.
Also, note that

(1) if $A \in \mathscr{D}(\mathbb{R}^p)$, then $L(A)$ has content and $c(L(A)) \geq 0$.

(2) suppose A, $B \in \mathscr{D}(\mathbb{R}^p)$ and $A \cap B = \emptyset$, then $L(A) \cap L(B) = \emptyset$ and $c(L(A \cup B)) = c(L(A) \cup L(B)) = c(L(A)) + c(L(B))$.

(3) if $x \in \mathbb{R}^p$ and $A \in \mathscr{D}(\mathbb{R}^p)$, then $L(x + A) = L(x) + L(A)$ and $c(L(x + A)) = c(L(A))$.
These imply that

(i) there exists a constant $m_L \geq 0$ such that $c(L(A)) = m_L c(A)$ for all $A \in \mathscr{D}(\mathbb{R}^p)$.

(ii) if L, $M \in \mathscr{L}(\mathbb{R}^p)$ are nonsingular maps and if $A \in \mathscr{D}(\mathbb{R}^p)$, since $m_{L\circ M}\, c(A) = c(L \circ M(A)) = m_L\, c(M(A)) = m_L m_M c(A)$, we have $m_{L\circ M} = m_L m_M$.

(iii) one can show that $m_L = |\det L|$ since every nonsingular $L \in \mathscr{L}(\mathbb{R}^p)$ is the composition of linear maps of the following three forms:

(a) $L_1(x_1, \ldots, x_p) = (\alpha x_1, x_2, \ldots, x_p)$ for some $\alpha \neq 0$;

(b) $L_2(x_1, \ldots, x_i, x_{i+1}, \ldots, x_p) = (x_1, \ldots, x_{i+1}, x_i, \ldots, x_p)$;

(c) $L_3(x_1, \ldots, x_p) = (x_1 + x_2, x_2, \ldots, x_p)$.

If $K_0 = [0,1) \times \cdots \times [0,1)$ in $\mathbb{R}^p$, then one can show that

$$\begin{aligned} m_{L_1} &= m_{L_1} c(K_0) = c(L_1(K_0)) = |\alpha| = |\det L_1|, \\ m_{L_2} &= m_{L_2} c(K_0) = c(L_2(K_0)) = 1 = |\det L_2|, \\ m_{L_3} &= m_{L_3} c(K_0) = c(L_3(K_0)) = 1 = |\det L_3|. \end{aligned}$$

Hence, we have $m_L = |\det L|$.

Lemma. Let $K \subset \mathbb{R}^p$ be a closed cell with center 0, Ω be an open set containing K and $\psi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$ and be injective. Suppose further that $J_\psi(x) \neq 0$ for $x \in K$ and that $\|\psi(x) - x\| \leq \alpha\|x\|$ for $x \in K$, and some constant $0 < \alpha < \dfrac{1}{\sqrt{p}}$. Then

$$(1 - \alpha\sqrt{p})^p \leq \frac{c(\psi(K))}{c(K)} \leq (1 + \alpha\sqrt{p})^p.$$

Proof. If the side length of K is $2r$ and if $x \in \partial K$, then we have

$$r \leq \|x\| \leq r\sqrt{p}.$$

This implies that

$$\|\psi(x) - x\| \leq \alpha\|x\| \leq \alpha r\sqrt{p},$$

i.e. $\psi(x)$ is within distance $\alpha r\sqrt{p}$ of $x \in \partial K$.
Note that $\mathbb{R}^p \setminus \partial K$ is a disjoint union of two nonempty open sets. Let
C_i be an open cell with center 0 and side length $2(1 - \alpha\sqrt{p})r$,
C_o be a closed cell with center 0 and side length $2(1 + \alpha\sqrt{p})r$.
Then we have $C_i \subset \psi(K) \subset C_o$, which implies that

$$(1-\alpha\sqrt{p})^p c(K) = (1-\alpha\sqrt{p})^p (2r)^p = c(C_i) \leq c(\psi(K)) \leq c(C_o) = (1+\alpha\sqrt{p})^p (2r)^p = (1+\alpha\sqrt{p})^p c(K).$$

The Jacobian Theorem. Let $\Omega \subseteq \mathbb{R}^p$ be open, $\phi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$, and be injective on Ω with $J_\phi(x) \neq 0$ for $x \in \Omega$. Let A have content and satisfy that $\bar{A} \subset \Omega$. If $\epsilon > 0$ is given, then there exists $\gamma > 0$ such that if K is a closed cell with center $x \in A$ and side length less than 2γ, then

$$|J_\phi(x)|(1 - \epsilon)^p \leq \frac{c(\phi(K))}{c(K)} \leq |J_\phi(x)|(1 + \epsilon)^p.$$

Proof. For each $x \in \Omega$, let $L_x = (d\phi(x))^{-1}$,
then $1 = \det(L_x \circ d\phi(x)) = (\det L_x)\,(\det d\phi(x))$,
it follows that $\det L_x = \dfrac{1}{\det d\phi(x)} = \dfrac{1}{J_\phi(x)}$.
Let Ω_1 be a bounded open subset of Ω such that

$$\bar{A} \subset \Omega_1 \subset \bar{\Omega}_1 \subseteq \Omega$$

and $\operatorname{dist}(A, \partial\Omega_1) = 2\delta > 0$.
Since $\phi \in C^1(\Omega)$, the entries in the standard matrix for L_x are continuous
There exists a constant $M > 0$ such that $\|L_x\|_{pp} \le M$ for all $x \in \Omega_1$.
Let $0 < \epsilon < 1$ be a constant.
Since $d\phi$ is uniformly continuous on Ω_1, there exists β with $0 < \beta < \delta$ such that if $x_1, x_2 \in \Omega_1$ and $\|x_1 - x_2\| \le \beta$, then $\|d\phi(x_1) - d\phi(x_2)\|_{pp} \le \epsilon/M\sqrt{p}$.
Now let $x \in A$ be given, hence if $\|z\| \le \beta$, then x, $x + z \in \Omega_1$.
Hence,

$$\begin{aligned}\|\phi(x+z) - \phi(x) - d\phi(x)(z)\| &\le \|z\| \sup_{0\le t\le 1} \|d\phi(x+tz) - d\phi(x)\|_{pp} \\ &\le \frac{\epsilon}{M\sqrt{p}}\|z\|.\end{aligned}$$

This implies that for a fixed $x \in A$ and for each $\|z\| \le \beta$ if we set

$$\psi(z) = L_x[\phi(x+z) - \phi(x)],$$

Then we have

$$\begin{aligned}\|\psi(z) - z\| &= \|L_x[\phi(x+z) - \phi(x) - d\phi(x)(z)]\| \\ &\le M\|z\| \sup_{0\le t\le 1} \|d\phi(x+tz) - d\phi(x)\|_{pp} \\ &\le \frac{\epsilon}{\sqrt{p}}\|z\| \text{ for } \|z\| \le \beta.\end{aligned}$$

If K_1 is any closed cell with center 0 and contained in the open ball with radius β, then

$$(1-\epsilon)^p \le \frac{c(\psi(K_1))}{c(K_1)} \le (1+\epsilon)^p.$$

It follows that if $K = x + K_1$ then K is a closed cell with center x and that $c(K) = c(K_1)$ and

$$c(\psi(K_1)) = |\det L_x| c(\phi(x + K_1) - \phi(x)) = \frac{1}{J_\phi(x)} c(\phi(K)).$$

This establishes the inequality

$$|J_\phi(x)|(1-\epsilon)^p \le \frac{c(\phi(K))}{c(K)} \le |J_\phi(x)|(1+\epsilon)^p.$$

for those closed cell K with center $x \in A$ and side length less than $2\gamma = 2\beta/\sqrt{p}$.
Change of Variables Theorem. Let $\Omega \subseteq \mathbb{R}^p$ be open, $\phi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$, and be injective on Ω, and $J_\phi(x) \ne 0$ for $x \in \Omega$. Suppose that A has content, $\bar{A} \subset \Omega$, and $f : \phi(A) \to \mathbb{R}$ is bounded and continuous.
Then $\int_{\phi(A)} f = \int_A (f \circ \phi)|J_\phi|$.
Example. Find $\iint_S (x^2 + y^2)dA$ if S be the region in the first quadrant bounded by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 4$. Setting $u = x^2 - y^2$, $v = xy$, we have $\iint_S (x^2+y^2)dA = \int_{v=1}^{v=3} \int_{u=1}^{u=4} \frac{1}{2}\, dudv = 3$.
Example. $\left(\int_0^\infty e^{-x^2}dx\right)^2 = \int_0^\infty \int_0^\infty e^{-x^2-y^2}dxdy = \int_0^{\pi/2} \int_0^\infty re^{-r^2}drd\theta = \frac{\pi}{2}$.
Summary.
(a) Let $A \subset \mathbb{R}^n$. Define what it means to say that A has content (or A is Jordan measurable).

(b) Let $A \subset \mathbb{R}^n$. Define the content $c(A)$ of A when A has content (or A is Jordan measurable).

(c) Let $A \subset \mathbb{R}^n$. Define what it means to say that A has content (or measure) $c(A)$ zero.

(d) Let A be a bounded subset of $\mathbb{R}^n$, and let f be a bounded function defined on A to $\mathbb{R}$. Define what it means to say that f is integrable on A. Give a class of A and a class of f from which $\int_A f$ exists.

(e) Let A be a bounded subset of $\mathbb{R}^n$ and let f be an integrable function defined on A to $\mathbb{R}$. Discuss the continuity of f on A.

(f) Let A be a bounded subset of $\mathbb{R}^n$ and let f be a continuous function defined on A to $\mathbb{R}$. Discuss the integrability of f on A?

(g) Let A be a bounded subset of $\mathbb{R}^n$, let f, g be integrable functions defined on A to $\mathbb{R}$, and let $a, b \in \mathbb{R}$. Discuss the integrability of $af + bg$ and fg on A.

(h) Let A be a bounded subset of $\mathbb{R}^n$ and let $\{f_n\}$ be a sequence of integrable function defined on A to $\mathbb{R}$. Assume that $f(x) = \lim f_n(x)$ exists for each $x \in A$. Discuss the integrability of f on A, and conditions on which the equality $\lim \int_A f_n = \int_A \lim f_n$ holds.

(i) Let $\Omega \subseteq \mathbb{R}^p$ be open and let $\phi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$. Suppose that A has content (or A is measurable), $\bar{A} \subset \Omega$. Discuss the measurability of $\phi(A)$.

(j) Let $\Omega \subseteq \mathbb{R}^p$ be open and let $\phi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$. Suppose that A has content zero and if $\bar{A} \subset \Omega$, discuss the measurability of $\phi(A)$.

(k) Let $L \in \mathscr{L}(\mathbb{R}^p) = \{B : \mathbb{R}^p \to \mathbb{R}^p \mid B = (b_{ij}) \text{ is an } p \times p \text{ matrix over } \mathbb{R}\}$ = the space of linear mappings on $\mathbb{R}^p$, and let $A \in \mathscr{D}(\mathbb{R}^p)$. Find $c(L(A))$.

(l) Let $\Omega \subseteq \mathbb{R}^p$ be open and let $\phi : \Omega \to \mathbb{R}^p$ belong to Class $C^1(\Omega)$. Then $d\phi(x) : \mathbb{R}^p \to \mathbb{R}^p$ is a linear mapping in $\mathscr{L}(\mathbb{R}^p)$. If $J_{\phi(x)} \neq 0$ for all $x \in \Omega$ and suppose that A has content (or A is measurable), $\bar{A} \subset \Omega$. Discuss the geometric meaning of $d\phi(x)(v)$ for each $v \in \mathbb{R}^p$, and the influence of $d\phi(x)$ on the volume element of A at x.